Name: Josef Ullrich
Class: ECE148
Instructor: Hua Lee

Assignment Eight

Introduction

In this assignment, I will design a digital highpass Butterworth filter using the bilinear transformation method. I will create a filter that attenuates lower digital frequencies while passing higher digital frequencies near $\Omega = \pm\pi$. My method will be to first convert the original digital highpass specifications into an analog lowpass Butterworth design, then I will find the required order and cutoff frequency, and finally transform the result back into the final digital highpass filter. I will plot the Amplitude Response of this filter to verify that my work converting the problem specifications into a filter was accurate.

Design Procedure

The goal of this assignment is to design a digital highpass Butterworth filter using the bilinear transformation. The filter should pass high digital frequencies through while attenuating lower digital frequencies. The design specifications I am provided are:

$$\Omega_p = \pm\frac{5\pi}{6}$$

$$\Omega_s = \pm\frac{2\pi}{3}$$

$$\alpha_{\max} = 0.5 \text{ dB}$$

$$\alpha_{\min} = 20 \text{ dB}$$

Since this is a highpass filter, frequencies $\Omega = \pm\pi$ should pass through the filter, while frequencies closer to $\Omega = 0$ should be removed.

My design procedure will be to first find the equivalent analog lowpass Buttworth filter specifications corresponding to the digital highpass filter. After I find this I will transform it back into the digital domain to obtain the desired highpass filter.

First, I use the bilinear transformation frequency mapping to convert the digital frequency, which is defined by:

$$\omega = \tan\left(\frac{\Omega}{2}\right)$$

Using this relationship, the digital passband and stopband frequencies are converted into analog frequencies:

$$\omega_p = \tan\left(\frac{\Omega_p}{2}\right)$$

$$\omega_s = \tan\left(\frac{\Omega_s}{2}\right)$$

This filter is still highpass, but I want to model it as a lowpass design problem, so I will use the frequency inversion:

$$\omega_{LP} = \frac{1}{\omega_{HP}}$$

This converts the highpass specifications into equivalent lowpass specifications:

$$\omega_{p,LP} = \frac{1}{\omega_p}$$

$$\omega_{s,LP} = \frac{1}{\omega_s}$$

Now I've gone from a digital highpass Butterworth design to an analog lowpass Butterworth design problem, which is easier to deal with because the magnitude formula, order equation, pole locations and other handy information is already defined for this form.

My next step will be to determine the Butterworth filter order n. The squared magnitude response of an analog lowpass Butterworth filter is:

$$|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_0}\right)^{2n}}$$

Where n is the filter order and $w0$ is the cutoff frequency. The order n controls how sharp the transition region is between the passband and the stopband

The Attenuation is just this magnitude converted to dB, which can be written as:

$$\alpha(\omega) = -20 \log_{10} |H(j\omega)|$$

Since the Butterworth formula uses $|H(j\omega)|^2$, I move the 2 in 2 * 10 = 20 over as an exponent to the magnitude and plug in the squared magnitude response:

$$10^{\alpha/10} = \frac{1}{|H(j\omega)|^2}$$

Now I plug in the Butterworth magnitude response which gives:

$$10^{\alpha/10} = 1 + \left(\frac{\omega}{\omega_0}\right)^{2n}$$

So:

$$10^{\alpha/10} - 1 = \left(\frac{\omega}{\omega_0}\right)^{2n}$$

Now I just use this equation at both the passband and stopband frequencies. At the passband edge:

$$10^{\alpha_{\max}/10} - 1 = \left(\frac{\omega_{p,LP}}{\omega_0}\right)^{2n}$$

At the stopband edge:

$$10^{\alpha_{\min}/10} - 1 = \left(\frac{\omega_{s,LP}}{\omega_0}\right)^{2n}$$

I want to eliminate ω_0, so I divide these two equations:

$$\frac{10^{\alpha_{\min}/10} - 1}{10^{\alpha_{\max}/10} - 1} = \left(\frac{\omega_{s,LP}}{\omega_{p,LP}}\right)^{2n}$$

I take $\log_{10}$ of both sides to simplify exponent:

$$\log_{10}\left(\frac{10^{\alpha_{\min}/10} - 1}{10^{\alpha_{\max}/10} - 1}\right) = 2n \log_{10}\left(\frac{\omega_{s,LP}}{\omega_{p,LP}}\right)$$

Now I solve for n:

$$n = \frac{1}{2}\frac{\log_{10}\left(\frac{10^{\alpha_{\min}/10}-1}{10^{\alpha_{\max}/10}-1}\right)}{\log_{10}\left(\frac{\omega_{s,LP}}{\omega_{p,LP}}\right)}$$

Since the filter order must be an integer, I will round this value up to the next whole number.

Once the filter order is found, I calculate the cutoff frequency ω_0. From the passband attenuation condition above, I get:

$$\omega_0 = \frac{\omega_{p,LP}}{\left(10^{\alpha_{\max}/10}-1\right)^{1/(2n)}}$$

After finding n and ω_0, I can use the standard Butterworth polynomial for the order that I find. For a Butterworth filter, I can use the known normalized lowpass denominator polynomial from the order n. I will call this normalized polynomial $B_n(s)$.

The normalized Butterworth transfer function has cutoff frequency $\omega_0 = 1$, so it can be written as:

$$H_{\text{norm}}(s) = \frac{1}{B_n(s)}$$

This normalized function has cutoff frequency $\omega_0 = 1$. Since my actual cutoff frequency will probably not be 1, I have to scale the normalized equation by replacing s with $\frac{s}{\omega_0}$. So then the analog lowpass Butterworth transfer function will become:

$$H(s) = \frac{1}{B_n\left(\frac{s}{\omega_0}\right)}$$

This gives me the analog lowpass Butterworth transfer function in terms of the order n and cutoff frequency ω_0. I will then plug in my actual provided values of n and ω_0, and write out the full function.

Now I have the analog lowpass transfer function, and all I need to do to convert it into a digital lowpass filter is to use the bilinear transformation:

$$s = \frac{1-z^{-1}}{1+z^{-1}}$$

Substituting this expression for s into the analog transfer function gives:

$$H(z) = H\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

This produces the equivalent lowpass digital filter. I then convert the lowpass digital filter into a highpass digital filter using the digital frequency transformation:

$$z^{-1} \to -z^{-1}$$

This gives the final digital highpass filter:

$$H(z) = H(z)\Big|_{z^{-1}\to -z^{-1}}$$

After finding $H(z)$, I will plot the amplitude response from $-\pi$ to $+\pi$ and verify the design by evaluating the frequency response at the original passband and stopband frequencies. At $\Omega_p = \pm 5\pi/6$, the attenuation should be no more than 0.5 dB. At $\Omega_s = \pm 2\pi/3$, the attenuation should be at least 20 dB, in accordance to the assignment directions.

Converted Analog Lowpass Design Specifications

Now I will plug in the digital highpass specifications and convert them into the corresponding analog lowpass Butterworth specifications, utilizing the equations explained in the above section.

The original digital highpass passband frequency is:

$$\Omega_p = \frac{5\pi}{6}$$

Using the bilinear transformation frequency mapping:

$$\omega = \tan\left(\frac{\Omega}{2}\right)$$

the analog highpass passband frequency is:

$$\omega_p = \tan\left(\frac{\Omega_p}{2}\right)$$

$$\omega_p = \tan\left(\frac{5\pi}{12}\right)$$

$$\omega_p \approx 3.7321$$

I use the same approach for the highpass stopband frequency is:

$$\Omega_s = \frac{2\pi}{3}$$

$$\omega_s = \tan\left(\frac{\Omega_s}{2}\right)$$

$$\omega_s = \tan\left(\frac{\pi}{3}\right)$$

$$\omega_s \approx 1.7321$$

I still have a highpass design because:

$$\omega_p > \omega_s$$

To convert this into an analog lowpass design, I use the frequency inversion:

$$\omega_{LP} = \frac{1}{\omega_{HP}}$$

The equivalent analog lowpass passband and stopband frequencies are:

$$\omega_{p,LP} = \frac{1}{\omega_p}$$

$$\omega_{p,LP} = \frac{1}{3.7321}$$

$$\omega_{p,LP} \approx 0.2679$$

$$\omega_{s,LP} = \frac{1}{\omega_s}$$

$$\omega_{s,LP} = \frac{1}{1.7321}$$

$$\omega_{s,LP} \approx 0.5774$$

I have successfully converted to a valid lowpass design because:

$$\omega_{p,LP} < \omega_{s,LP}$$

To summarize this section I now have:

$$\omega_{p,LP} \approx 0.2679$$

$$\omega_{s,LP} \approx 0.5774$$

With required specifications:

$$\alpha_{\max} = 0.5 \text{ dB}$$

$$\alpha_{\min} = 20 \text{ dB}$$

Now I will use these specifications to find the order and cutoff frequency of the analog lowpass Butterworth filter.

Order and Cutoff Frequency of the Analog Lowpass Butterworth Filter

From the previous section I found:

$$\omega_{p,LP} \approx 0.2679$$

$$\omega_{s,LP} \approx 0.5774$$

$$\alpha_{\max} = 0.5 \text{ dB}$$

$$\alpha_{\min} = 20 \text{ dB}$$

I can now find the required order n of the analog lowpass Butterworth filter. I use the order equation derived in the design procedure:

$$n = \frac{1}{2}\frac{\log_{10}\left(\frac{10^{\alpha_{\min}/10}-1}{10^{\alpha_{\max}/10}-1}\right)}{\log_{10}\left(\frac{\omega_{s,LP}}{\omega_{p,LP}}\right)}$$

Plug in:

$$n = \frac{1}{2}\frac{\log_{10}\left(\frac{10^{20/10}-1}{10^{0.5/10}-1}\right)}{\log_{10}\left(\frac{0.5774}{0.2679}\right)}$$

$$n \approx 4.36$$

Filter order has to be an integer, so I round up:

$$n = 5$$

Now that I have the filter order, I can calculate the cutoff frequency ω_0. From the passband attenuation condition (also derived in design procedure):

$$\omega_0 = \frac{\omega_{p,LP}}{\left(10^{\alpha_{\max}/10} - 1\right)^{1/(2n)}}$$

Plug in values from above:

$$\omega_0 = \frac{0.2679}{\left(10^{0.5/10} - 1\right)^{1/(2(5))}}$$

$$\omega_0 \approx 0.3307$$

So the corresponding analog lowpass Butterworth filter has order:

$$n = 5$$

and cutoff frequency:

$$\omega_0 \approx 0.3307$$

Analog Lowpass Butterworth Transfer Function

From the previous section, I found that the analog lowpass Butterworth filter has order:

$$n = 5$$

and cutoff frequency:

$$\omega_0 \approx 0.3307$$

Since this is a 5th-order Butterworth filter, I use the normalized 5th-order Butterworth denominator polynomial, pulling the coefficients from Wikipedia:

$$B_5(s) = s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1$$

This normalized polynomial corresponds to a cutoff frequency of $\omega_0 = 1$. Since the cutoff frequency for this filter is $\omega_0 \approx 0.3307$, I have to scale this equation, by replacing s with $\frac{s}{\omega_0}$. The Butterworth transfer function is defined as:

$$H(s) = \frac{1}{B_5\left(\frac{s}{\omega_0}\right)}$$

Now I plug in the known 5th order polynomial:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_0}\right)^5 + 3.2361\left(\frac{s}{\omega_0}\right)^4 + 5.2361\left(\frac{s}{\omega_0}\right)^3 + 5.2361\left(\frac{s}{\omega_0}\right)^2 + 3.2361\left(\frac{s}{\omega_0}\right) + 1}$$

Multiplying the numerator and denominator by ω_0^5, this turns into:

$$H(s) = \frac{\omega_0^5}{s^5 + 3.2361\omega_0 s^4 + 5.2361\omega_0^2 s^3 + 5.2361\omega_0^3 s^2 + 3.2361\omega_0^4 s + \omega_0^5}$$

Now I substitute:

$$\omega_0 \approx 0.3307$$

Substituting this into the denominator gives:

$$H(s) = \frac{0.0040}{s^5 + 1.0702s^4 + 0.5728s^3 + 0.1895s^2 + 0.0388s + 0.0040}$$

Equivalent Lowpass Digital Transfer Function

I am starting this section with the analog lowpass Butterworth transfer function found in the previous section:

$$H(s) = \frac{0.0040}{s^5 + 1.0702s^4 + 0.5728s^3 + 0.1895s^2 + 0.0388s + 0.0040}$$

Now I want to convert this analog lowpass filter into a digital lowpass filter. To do so I use the bilinear transformation:

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

I plug this into the analog transfer function and multiply the numerator and denominator by $(1 + z^{-1})^5$. The result of these 2 combined steps is:

$$\hat{H}(z) = \frac{0.0040(1 + z^{-1})^5}{\begin{aligned}&(1 - z^{-1})^5 + 1.0702(1 - z^{-1})^4(1 + z^{-1}) \\ &+ 0.5728(1 - z^{-1})^3(1 + z^{-1})^2 \\ &+ 0.1895(1 - z^{-1})^2(1 + z^{-1})^3 \\ &+ 0.0388(1 - z^{-1})(1 + z^{-1})^4 \\ &+ 0.0040(1 + z^{-1})^5\end{aligned}}$$

To ensure correctness, I use a polynomial calculator for this expansion and simplification. The result is:

$$\hat{H}(z) = \frac{\begin{aligned}&0.0014 + 0.0069z^{-1} + 0.0138z^{-2} \\ &+ 0.0138z^{-3} + 0.0069z^{-4} + 0.0014z^{-5}\end{aligned}}{\begin{aligned}&1 - 2.9422z^{-1} + 3.7338z^{-2} \\ &- 2.4807z^{-3} + 0.8541z^{-4} - 0.1210z^{-5}\end{aligned}}$$

Therefore, the equivalent lowpass digital transfer function is:

$$\hat{H}(z) = \frac{0.0014 + 0.0069z^{-1} + 0.0138z^{-2} + 0.0138z^{-3} + 0.0069z^{-4} + 0.0014z^{-5}}{1 - 2.9422z^{-1} + 3.7338z^{-2} - 2.4807z^{-3} + 0.8541z^{-4} - 0.1210z^{-5}}$$

This is still a lowpass filter. In the next section, I will transform it into a highpass filter.

Highpass Digital Transfer Function

From the previous section, I found the equivalent lowpass digital filter as:

$$\hat{H}(z) = \frac{0.0014 + 0.0069z^{-1} + 0.0138z^{-2} + 0.0138z^{-3} + 0.0069z^{-4} + 0.0014z^{-5}}{1 - 2.9422z^{-1} + 3.7338z^{-2} - 2.4807z^{-3} + 0.8541z^{-4} - 0.1210z^{-5}}$$

This filter is still a lowpass digital filter. To convert it into the desired highpass digital filter, I use the digital frequency transformation:

$$z^{-1} \to -z^{-1}$$

Applying this substitution to the lowpass transfer function gives me the final digital highpass Butterworth filter:

$$H(z) = \frac{0.0014 - 0.0069z^{-1} + 0.0138z^{-2} - 0.0138z^{-3} + 0.0069z^{-4} - 0.0014z^{-5}}{1 + 2.9422z^{-1} + 3.7338z^{-2} + 2.4807z^{-3} + 0.8541z^{-4} + 0.1210z^{-5}}$$

Frequency-Response Plot

Using the final digital highpass Butterworth filter $H(z)$, I plotted the amplitude response from $-\pi$ to $+\pi$.

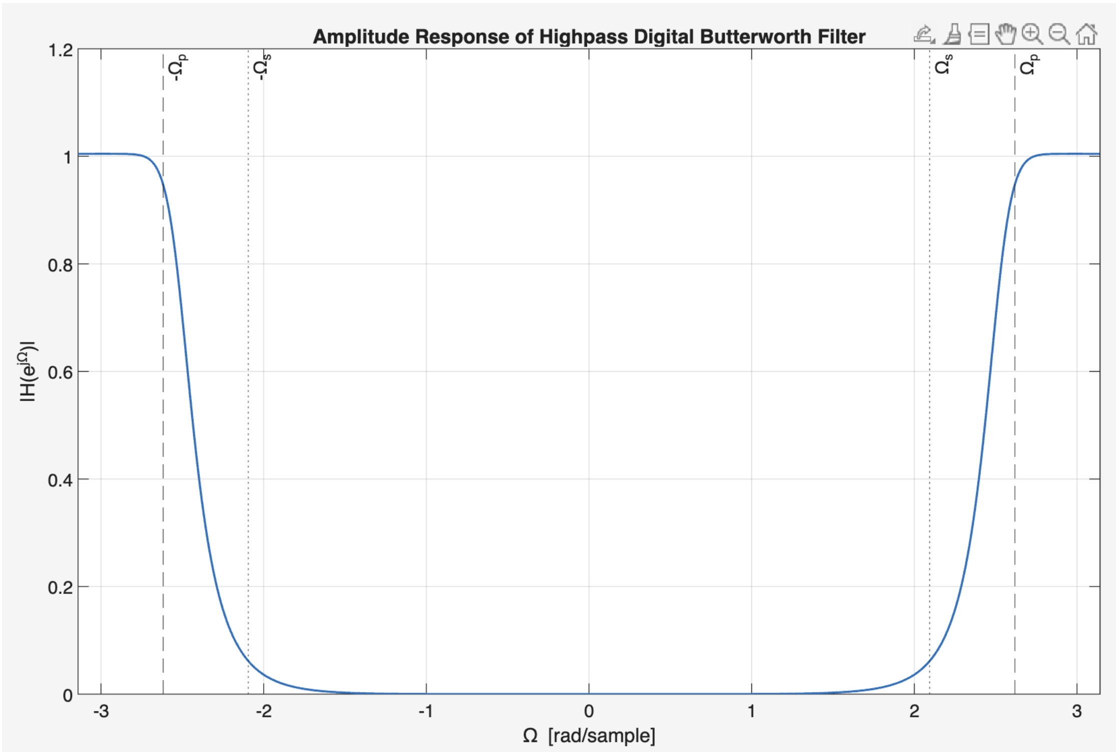

Figure 1: Amplitude response of the final digital highpass Butterworth filter from $-\pi$ to $+\pi$.

From this plot, I can tell that I have achieved the desired assignment goal. The magnitude is close to zero near $\Omega = 0$, so low digital frequencies are attenuated. The magnitude approaches one near $\Omega = \pm\pi$, so high digital frequencies are passed. The dashed vertical lines show the passband frequencies $\Omega_p = \pm 5\pi/6$, and the dotted vertical lines show the stopband frequencies $\Omega_s = \pm 2\pi/3$. As expected I see that these lines are the transition points of this highpass filter, as desired.

Design Verification

To verify the final filter design, I evaluated the frequency response of the highpass digital filter at the original passband and stopband frequencies. The passband and stopband frequencies are:

$$\Omega_p = \frac{5\pi}{6} \approx 2.6180$$

$$\Omega_s = \frac{2\pi}{3} \approx 2.0944$$

Using Matlab, I evaluated $H(e^{j\Omega})$ at both of these frequencies and converted the magnitudes into attenuation values using:

$$\alpha(\Omega) = -20 \log_{10} |H(e^{j\Omega})|$$

Matlab code used (also for Frequency-Response Plot):

```
% hw8: Filter plot and verification
% final highpass digital filter coefficients
b = [0.0014 -0.0069 0.0138 -0.0138 0.0069 -0.0014];
a = [1 2.9422 3.7338 2.4807 0.8541 0.1210];

% frequency response from -pi to pi
N = 2048;

[H, w] = freqz(b, a, N, 'whole');

% shift from [0, 2pi] to [-pi, pi]
H_shift = fftshift(H);
w_shift = w - pi;

% plot
figure;
plot(w_shift, abs(H_shift), 'LineWidth', 1.2);
xlabel('\Omega [rad/sample]');
ylabel('|H(e^{j\Omega})|');
title('Amplitude Response of Highpass Digital Butterworth Filter');
grid on;
xlim([-pi pi]);
ylim([0 1.2]);

% mark passband and stopband frequencies
hold on;
xline(5*pi/6, '--', '\Omega_p');
xline(-5*pi/6, '--', '-\Omega_p');
xline(2*pi/3, ':', '\Omega_s');
xline(-2*pi/3, ':', '-\Omega_s');
```

```
hold off;

% save figure
saveas(gcf, 'highpass_frequency_response.png');

% now I verify the attentuation to make sure I'm meeting specifications
Omega_p = 5*pi/6;
Omega_s = 2*pi/3;

% find H at these frequencies
[Hcheck, Wcheck] = freqz(b, a, [Omega_p Omega_s]);

Hp = Hcheck(1);
Hs = Hcheck(2);

att_p = -20*log10(abs(Hp));
att_s = -20*log10(abs(Hs));

fprintf('Passband magnitude at Omega_p = %.4f rad/sample: %.4f\n', Omega_p, abs(Hp));
fprintf('Passband attenuation at Omega_p = %.4f rad/sample: %.4f dB\n', Omega_p, att_p);

fprintf('Stopband magnitude at Omega_s = %.4f rad/sample: %.4f\n', Omega_s, abs(Hs));
fprintf('Stopband attenuation at Omega_s = %.4f rad/sample: %.4f dB\n', Omega_s, att_s);
```

The Matlab output was:

$$|H(e^{j\Omega_p})| = 0.9473$$

$$\alpha_p = 0.4701 \text{ dB}$$

$$|H(e^{j\Omega_s})| = 0.0616$$

$$\alpha_s = 24.2057 \text{ dB}$$

The passband attenuation requirement is:

$$\alpha_p \leq 0.5 \text{ dB}$$

Since:

$$0.4701 \text{ dB} \leq 0.5 \text{ dB}$$

the passband requirement is satisfied.

The stopband attenuation requirement is:

$$\alpha_s \geq 20 \text{ dB}$$

Since:

$$24.2057 \text{ dB} \geq 20 \text{ dB}$$

the stopband requirement is also satisfied. Therefore, my Butterworth filter satisfies all problem specifications, verifying all my above work.

Conclusion

In conclusion, I was able to achieve the desired Butterworth filter: The final frequency-response verification showed 0.4701 dB of passband attenuation and 24.2057 dB of stopband attenuation, so the filter satisfies both design requirements. This assignment taught me the process of taking filter specifications and mathematically converting them into the design of a filter, which involved spending time in the analog, discrete, highpass, and lowpass domains. It was interesting to me to see where each of these pieces fit into the overall system design.